Recall: the definition of a vector space over a field.

Definition: (subspace)

Let V be a vector space over $\mathbb{F}$. A nonempty subset $W \subseteq V$ is called a **subspace** of V if W is also a vector space over $\mathbb{F}$ with the same operations of addition and scalar multiplication as V.

But we will not want to check all of the vector space axioms!

Subspace Test: Let V be a vector space over $\mathbb{F}$. $W \subseteq V$ is a subspace of V if and only if $\forall\, u, v \in W$ and $\alpha \in \mathbb{F}$,

1) $0_V \in W$

2) $u - v \in W$

3) $\alpha \cdot u \in W$

Remark: You can replace 1) with "W is nonempty" and combine 2) and 3) into "$\alpha \cdot u - v \in W$".

Example 1: (upper triangular matrices)

Recall $M_n(\mathbb{F})$ for $\mathbb{F}$ a field and $n \in \mathbb{N}$, the $n \times n$ matrices over $\mathbb{F}$. Let $W \subseteq M_n(\mathbb{F})$ be the upper triangular matrices:

$$(\alpha_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{F})$$

is in W if and only if

$$\boxed{\alpha_{i,j} = 0_{\mathbb{F}} \text{ whenever } i > j.}$$

For $n=2$: $\begin{pmatrix} \alpha & \beta \\ 0_{\mathbb{F}} & \gamma \end{pmatrix}$

is a typical element in W.

$n=3$: $\begin{pmatrix} \alpha & \beta & \gamma \\ 0_{\mathbb{F}} & \varepsilon & \delta \\ 0_{\mathbb{F}} & 0_{\mathbb{F}} & \Omega \end{pmatrix}$

Use subspace test.

1) $0_V \in W$: 0_V

has $\alpha_{i,j} = 0_{\mathbb{F}}$ for all

(i,j), not just $i > j$.

so $0_V \in W$.

2) Let $u, v \in W$.

$u = (u_{i,j})_{i,j=1}^{n}$

$v = (v_{i,j})_{i,j=1}^{n}$

$v_{i,j} = u_{i,j} = 0$ when $i > j$.

Then

$(u-v)_{i,j} = u_{i,j} - v_{i,j}$

$= 0_{\mathbb{F}}$ when $i>j$

3) Let $v = (v_{i,j})_{i,j=1}^{n} \in W$

and $\alpha \in \mathbb{F}$.

Then $v_{i,j} = 0 \ \forall \ i > j$,

so

$$(\alpha \cdot v)_{i,j} = \alpha \cdot v_{i,j}$$

$$= 0 \text{ when } i > j$$

This shows W is a subspace of V.

Remark: Using the same argument and changing notation, you can show the lower triangular matrices are also a subspace of $M_n(\mathbb{F})$.

Example 2: (c_0 in c)

c will denote the real (or complex, if you like) vector space of convergent sequences of real (complex) numbers. c_0 denotes the subset of sequences that converge to zero.

Subspace Test

1) $0_V \in c_0$!

$$0_V = (0, 0, 0, \dots)$$

the sequence of all zeros, which certainly converges to zero!

2) Let $u=(u_i)_{i=1}^{\infty}$

and $v=(v_i)_{i=1}^{\infty}$

be elements of c_0.

Then $\lim_{i\to\infty} u_i = 0 = \lim_{i\to\infty} v_i$.

Therefore,

$$\lim_{i\to\infty}(u_i+v_i) = \lim_{i\to\infty} u_i + \lim_{i\to\infty} v_i$$

$$= 0+0 = 0$$

3) Let $u = (u_i)_{i=1}^{\infty} \in c_0$

and let $\alpha \in \mathbb{R}$ (or $\mathbb{C}$).

Then

$$\lim_{i\to\infty} (\alpha u_i) = \alpha \lim_{i\to\infty} u_i$$

$$= \alpha \cdot 0 = 0$$

since $\lim_{i\to\infty} u_i = 0$.

Therefore c_0 is a subspace of c.

Example 3: (not a subspace)

$S \subseteq \mathbb{R}^2$, $p(x) = x(x-1)(x+1)$

$$S = \{(x,0) \mid p(x) = 0\}$$

Is S a subspace of $\mathbb{R}^2$ (considered as a vector space over $\mathbb{R}$)?

Note $(0,0) \in S$, so

S contains the additive identity of $\mathbb{R}^2$.

Let's show vector addition fails.

The elements of S are $(0,0)$, $(1,0)$, and $(-1,0)$.

Then $(1,0) - (-1,0) = (2,0) \notin S$.

So S is not a subspace of $\mathbb{R}^2$.

Proof of Subspace test:

$\Rightarrow$ (if and only if statements require two directions)

Suppose W is a subspace of V. Since W is nonempty, $\exists\, x \in W$.

Then $(-1_{\mathbb{F}})x = -x \in W$

Since W is a vector space.

Since W is a vector space,

$$x + (-x) = 0_V \in W$$

Now let $x, y \in W$.

Then again $-y = (-1_{\mathbb{F}}) y \in W$,

So $x - y = x + (-y) \in W$

since W is a vector space.

Since W is a vector space over $\mathbb{F}$, we have $\alpha \cdot x \in W$ $\forall\ x \in W, \alpha \in \mathbb{F}$ by definition of a vector space.

Done with one direction!

$\Leftarrow$ Suppose W satisfies the conclusion of the subspace test, i.e. $0_V \in W$, $x-y \in W \ \forall \ x,y \in W$, $\alpha x \in W \ \forall \ x \in W, \alpha \in \mathbb{F}$.

Since $(-1_{\mathbb{F}})y = -y \in W$,

$x + y = x - (-y) \in W$,

and $\alpha x \in W$ is immediate.

Therefore the binary operations of $\mathcal{V}$ restrict to those of W.

We get as a consequence (since we already know these facts for V) that distributitivity and associativity hold for scalar multiplication over addition.

Since "+" is commutative on V, it is commutative on W. Associativity of "+" on W follows from associativity on V.

Now if $x \in W$, then $-x = (-1_{\mathbb{F}})x \in W$ since $\alpha x \in W \ \forall \ x \in W, \alpha \in \mathbb{F}$.

We've already shown $0_V \in W$.

We have then shown that $(W, +)$ is an abelian group. Since $1_{\mathbb{F}} x = x \ \forall x \in V$, if $x \in W \subseteq V$, this still holds. Therefore W is a vector space over $\mathbb{F}$! $\square$